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On the stability of viscous flow between rotating cylinders

Part 1. Asymptotic analysis

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The stability of Couette flow is discussed in the case in which the cylinders rotate in opposite directions by an asymptotic method in which the Taylor number is treated as a large parameter. On assuming the principle of exchange of stabilities to hold, the problem is then governed by a sixth-order differential equation with a simple turning point. It is shown how the solutions of this equation can be represented asymptotically in terms of the solutions of the comparison equation $y^{\mathrm{vi}} = xy$. The solutions of this comparison equation have recently been tabulated and we thus have an explicit representation of the solution of the stability problem in terms of tabulated functions. Detailed results for the critical Taylor number and wave-number at the onset of instability and the associated eigenfunctions are given for the limiting case $\mu \to -\infty$, where $\mu = \Omega_2/\Omega_1$, and Ω_1 and Ω_2 are the angular velocities of the inner and outer cylinders respectively. In this limiting case it is found that there exists an infinite number of cells between the cylinders, but that the amplitude of the secondary motion in all but the innermost cell is small.

1. Introduction

The problem of the stability of viscous flow between rotating cylinders was first successfully treated both theoretically and experimentally by G. I. Taylor in 1923, and it has since been considered by a number of other workers. One of the chief difficulties of the problem arises from the fact that the character of the eigenvalue problem to which one is led depends very markedly on whether the cylinders rotate in the same or in opposite directions, i.e. it depends on the sign of $\Omega_2/\Omega_1 = \mu$ say, where Ω_1 and Ω_2 are the angular velocities of the inner and outer cylinders respectively. In recent years adequate methods have been developed for dealing with the problem when μ is positive, or when μ is only moderately negative. For large negative values of μ, however, these methods become inadequate, and an essentially different approach must be adopted. The purpose of the present paper then is to describe an asymptotic method which is especially suited for dealing with the case when $\mu < 0$.

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2. The governing equations

Let (r, ϕ, z) be cylindrical co-ordinates with the axis of the cylinders along the z-axis, and let R_1 and R_2 denote the radii of the inner and outer cylinders respectively. The equations of motion then allow the stationary solution

$$\Omega(r) = A + B/r^2 \tag{1}$$

for the angular velocity at a distance r from the axis of rotation. The constants A and B are given by

$$A = -\Omega_1(\eta^2-\mu)/(1-\eta^2) \quad \text{and} \quad B = \Omega_1 R_1^2(1-\mu)/(1-\eta^2), \tag{2}$$

where $\eta = R_1/R_2$.

When the difference in radii of the two cylinders is small compared with their mean radius, i.e. when

$$d = R_2 - R_1 \ll \tfrac{1}{2}(R_2+R_1), \tag{3}$$

the angular velocity distribution can be approximated by

$$\Omega(r) \simeq \Omega_1[1-(1-\mu)\zeta], \tag{4}$$

where

$$\zeta = (r-R_1)/d. \tag{5}$$

When this approximation is not made, the resulting eigenvalue problem is of considerably greater difficulty and we have not attempted to extend the present method of solution to that case. Among the work dealing with the finite-gap problem, however, we may mentioned the papers by Chandrasekhar (1958) and Chandrasekhar & Elbert (1962) which deal with the particular case $\eta = \frac{1}{2}$, the papers by Miss Steinman (1956) and Witting (1958) which are based on an expansion of the solution in powers of d/R_1, and the papers by Kirchgässner (1961) and Walowit, Tsao & Di Prima (1964) in which the only essential limitation is that $-\mu$ is not too large.

If the velocity distribution given by equation (4) is subjected to a rotationally symmetric perturbation whose t and z dependence is of the form

$$\exp(pt+ikz), \tag{6}$$

then the linearized equation for v, the amplitude of the azimuthal component of the perturbation velocity, can be written in the non-dimensional form (cf. Lin 1955, p. 18)

$$(D^2-a^2-\sigma\sqrt{T})^2(D^2-a^2)v = -a^2T[1-(1-\mu)\zeta]v, \tag{7}$$

where the Taylor number and the amplification factor σ are given by

$$T = -(4A\Omega_1/\nu^2)d^4 \quad \text{and} \quad \sigma = p/(-4A\Omega_1)^{\frac{1}{2}}. \tag{8}$$

In equation (7), D stands for d/dz and $a = kd$. Within the framework of the small-gap approximation (3) the parameters defined by (8) can be written in the alternative forms

$$T = 2(1-\mu)\left(\frac{\Omega_1 R_1 d}{\nu}\right)^2 \frac{d}{R_1} \quad \text{and} \quad \sigma = \frac{p}{\Omega_1}\left(\frac{d}{2(1-\mu)R_1}\right)^{\frac{1}{2}}. \tag{9}$$

It may be noted that in the derivation of equation (7), the amplitude of the radial component of the perturbation velocity, u, is found to be proportional to $(D^2-a^2-\sigma\sqrt{T})\,v$. Equation (7) must be solved subject to the boundary conditions

$$v = (D^2-a^2-\sigma\sqrt{T})\,v = D(D^2-a^2-\sigma\sqrt{T})\,v = 0 \tag{10}$$

at $\zeta = 0$ and $\zeta = 1$.

For assigned values of μ, the eigenvalue problem defined by the equation (7) and the boundary conditions (10) leads to a characteristic equation of the form

$$F(a, T, \sigma) = 0. \tag{11}$$

If we let $\sigma = \sigma_r + i\sigma_i$, then the condition for neutral stability is that $\sigma_r = 0$ and this condition leads to a relationship between a and T. The minimum value of T obtained from this relation, together with the corresponding values of a and σ_i, define the conditions under which instability will first occur. When σ_i vanishes at the onset of instability, the 'principle of exchange of stabilities' is said to hold, and the resulting neutral mode is in the form of a steady secondary motion. If σ_i does not vanish, however, then we are dealing with the case of overstability and the resulting neutral mode has an oscillatory character.

From an analysis of the inviscid form of equation (7), obtained by formally allowing T to become infinite in that equation, it has been suggested (Reid 1960) that while the principle of exchange of stabilities may hold when the cylinders rotate in the same direction, overstable modes may also appear when they rotate in opposite directions. In the present paper we will treat only the convective modes of instability, i.e. we will continue to assume that the principle of exchange of stabilities is valid even for negative values of μ. It would appear, however, that the method to be described can be adapted to the study of the overstable modes as well.

On the assumption then that $\sigma \equiv 0$ describes the state of marginal stability, equation (7) becomes

$$(D^2-a^2)^3\,v = -a^2T[1-(1-\mu)\,\zeta]\,v. \tag{12}$$

with the boundary conditions

$$v = (D^2-a^2)\,v = D(D^2-a^2)\,v = 0 \tag{13}$$

at $\zeta = 0$ and $\zeta = 1$. When the cylinders rotate in the same direction, the coefficient of v on the right-hand side of equation (7) does not change sign, and under these circumstances one can, to a good approximation, simply replace this coefficient by its average value to obtain

$$T_c \approx 1708/\tfrac{1}{2}(1+\mu) \quad \text{and} \quad a_c \approx 3{\cdot}12. \tag{14}$$

Strictly speaking, these results are only valid in the limit $\mu \to 1$, but the error is of the order of ϵ^2, where

$$\epsilon = 2(1-\mu)/(1+\mu), \tag{15}$$

with a coefficient that is numerically very small (cf. Chandrasekhar 1961). The results (14) thus provide, as was first noted by Taylor, a good approximation for $0 < \mu \leqslant 1$. Since the problem is not self-adjoint, however, it may also possess complex eigenvalues for $\mu < 1$. This can also be seen from the fact that the expansion of the solution of equation (12) in powers of ϵ would appear to be asymptotic rather than convergent and thus cannot prove that all of the eigenvalues are positive or even real for any $\epsilon > 0$.

When the cylinders rotate in opposite directions, a transition point occurs in equation (12) at $\zeta_c = 1/(1-\mu)$ and, on the assumption that real eigenvalues exist, the equation would then admit solutions for both positive and negative values of T. Only positive values of T are, of course, physically relevant. On Rayleigh's criterion it is only the fluid lying between the inner cylinder and the nodal point that is dynamically unstable, and this result suggests that when the cylinders rotate in opposite directions the relevant length scale is no longer the gap width, but rather the distance from the inner cylinder to the nodal point. For $\mu < 0$, it is convenient, therefore, to renormalize the problem by writing

$$z = (1-\mu)\,\zeta, \tag{16}$$

so that the inner cylinder is located at $z = 0$, and the nodal point is fixed at $z = 1$. The wave-number and the Taylor number can also be conveniently renormalized by letting

$$\alpha = a/(1-\mu) \quad \text{and} \quad \tau = T/(1-\mu)^4. \tag{17}$$

Equation (12) then becomes

$$(D^2-\alpha^2)^3\, v = -\alpha^2\tau(1-z)\, v, \tag{18}$$

where D now stands for d/dz, and the boundary conditions (13) become

$$v = (D^2-\alpha^2)\, v = D(D^2-\alpha^2)\, v = 0 \tag{19}$$

at $z = 0$ and $z = 1-\mu$.

3. Discussion of previous work

One of the most powerful methods for dealing with the eigenvalue problem defined by equations (12) and (13) is the Fourier-expansion technique developed by Chandrasekhar (1954, 1961). This method has proved to be entirely satisfactory for positive or for moderately negative values of μ. But for values of μ more negative than about -3, the method becomes increasingly difficult to apply. This difficulty can be traced to the fact that as μ takes on large negative values the required eigenfunction has a highly damped oscillatory character (even for the lowest mode) over a large part of the interval and, as a result, an increasingly large number of terms in the Fourier expansion is required to represent it adequately. Fortunately, the values of τ_c and α_c become nearly independent of μ as soon as μ becomes more negative than about -2. Chandrasekhar was able, therefore, to estimate the limiting values of τ_c and α_c as $\mu \to -\infty$, with the results

$$\tau_c \to 1180 \quad \text{and} \quad \alpha_c \to 2{\cdot}03 \quad \text{as} \quad \mu \to -\infty. \tag{20}$$

The problem has also been discussed by Di Prima (1955) using a Galerkin method. In his analysis he employed equations equivalent to (18) and (19) and, for $\mu \to -\infty$, obtained the results

$$\tau_c \to 1075 \quad \text{and} \quad \alpha_c \to 2{\cdot}125 \quad \text{as} \quad \mu \to -\infty. \tag{21}$$

While the trial functions used in this calculation lead to adequate estimates for τ_c and α_c as $\mu \to -\infty$, they do not have the correct behaviour for large values of z and hence cannot provide an adequate approximation to the corresponding eigenfunctions.

An asymptotic method of integrating equation (12) has been described by Meksyn (1946, 1961) on the assumption that the wave-number a is a large parameter.† On the further assumption that T/a^4 is of order unity, he obtained WKB solutions of the form

$$v = \left(\frac{\partial f}{\partial V}\right)^{-\frac{1}{2}} \exp\left[a \int V(\zeta)\, d\zeta\right], \tag{22}$$

where
$$f(V,\zeta) = (V^2-1)^3 + T[1-(1-\mu)\,\zeta]/a^4, \tag{23}$$

and $V(\zeta)$ is a root of the equation $f(V,\zeta) = 0$. The zeros of $\partial f/\partial V$ are given by the roots of the equation
$$6V(V^2-1)^2 = 0, \tag{24}$$

and it is seen that in the representation (22) there are in general two critical points located at
$$\zeta/\zeta_c = 1 \quad \text{and} \quad \zeta/\zeta_c = 1 - a^4/T. \tag{25}$$

When the cylinders rotate in opposite directions both of these critical points lie in the interval $0 < \zeta < 1$, and it would, therefore, be necessary to obtain the continuation of the solutions through both of them. Since these critical points are close together, Meksyn assumes that they coincide, and then chooses to deal with the single critical point at $\zeta/\zeta_c = 1 - a^4/T$. His analysis leads to the results

$$\tau_c \to 1132 \quad \text{and} \quad \alpha_c \to 2 \quad \text{as} \quad \mu \to -\infty, \tag{26}$$

but it does not lead to a convenient representation for the corresponding eigenfunctions.

In the definition of the Taylor number T there occurs a factor $1/\nu^2$ and, as in other problems of hydrodynamic stability, this fact suggests that T (or τ) be considered a large parameter. On the assumption, therefore, that α is of order unity and that T is large, it will be shown that the solution of the present problem can be represented asymptotically in terms of the solutions of the comparison equation
$$y^{\mathrm{vi}} = xy. \tag{27}$$

This method avoids most of the difficulties associated with Meksyn's approach and, in addition, does provide a convenient representation for the eigenfunctions.

4. The method of solution

The equation (18) with which we must deal is a linear sixth-order equation with a simple turning point at $z = 1$. Equations of this type have been discussed by Langer (1960), and while the method to be presented here has many similarities with Langer's method, it differs from his in some details. Consider first the change of variable
$$x = \lambda(z-1). \tag{28}$$

† The quantity h which appears in Meksyn's analysis is related to the parameters defined here by the relation $hd = (1-\mu)\, T/a^4$.

If we identify λ with $(\alpha^2\tau)^{\frac{1}{7}}$, then equation (18) becomes

$$(D^2-\alpha^2/\lambda^2)^3\,v = xv, \tag{29}$$

where D now stands for d/dx, and the boundary conditions (19) become

$$v = (D^2-\alpha^2/\lambda^2)\,v = D(D^2-\alpha^2/\lambda^2)\,v = 0 \tag{30}$$

at $x = -\lambda$ and $x = -\lambda\mu$.

The general solution of equation (29) can be written in the usual form

$$v(x) = \sum_{i=1}^{6} C_i\,v_i(x), \tag{31}$$

where $\{v_i(x)\}$ is a fundamental set of solutions of equation (29). The boundary conditions (30) then lead to a set of six homogeneous equations in the C's, and the requirement that they not vanish identically then leads to the characteristic equation

$$\begin{vmatrix} v_i(-\lambda) \\ (D^2-\alpha^2/\lambda^2)\,v_i(-\lambda) \\ D(D^2-\alpha^2/\lambda^2)\,v_i(-\lambda) \\ v_i(-\lambda\mu) \\ (D^2-\alpha^2/\lambda^2)\,v_i(-\lambda\mu) \\ D(D^2-\alpha^2/\lambda^2)\,v_i(-\lambda\mu) \end{vmatrix} = 0, \tag{32}$$

where $i = 1, 2, \ldots, 6$.

From the form of equation (29) it is clear that it is the parameter $\lambda = (\alpha^2\tau)^{\frac{1}{7}}$ that must be considered large rather than τ itself. The quantity $(\alpha^2\tau)^{\frac{1}{7}}$ thus plays a role in the present problem analogous to the quantity $(\alpha R)^{\frac{1}{3}}$ in the asymptotic theory of the Orr–Sommerfeld equation. From the values of α_c and τ_c quoted in the previous section, it follows that $\lambda_c \approx 3{\cdot}4$ for $\mu \to -\infty$. While this value of λ might appear to be somewhat small, it will be shown that it is in fact sufficiently large for the present purposes.

On the assumption then that λ is a large parameter, we will now derive a formal asymptotic solution of equation (29) in the form

$$v_i(x) = \sum_{n=0}^{\infty} \lambda^{-2n}\,v_i^{(2n)}(x), \tag{33}$$

where $i = 1, 2, \ldots, 6$. On substituting this representation for $v_i(x)$ into equation (29) and requiring the resulting equation to be satisfied to various orders in $1/\lambda^2$, we obtain the sequence of equations

$$(D^6-x)\,v_i^{(0)} = 0, \tag{34a}$$

$$(D^6-x)\,v_i^{(2)} = 3\alpha^2D^4v_i^{(0)}, \tag{34b}$$

$$(D^6-x)\,v_i^{(4)} = 3\alpha^2D^4v_i^{(2)} - 3\alpha^4D^2v_i^{(0)}, \tag{34c}$$

and, for $n \geqslant 3$,

$$(D^6-x)\,v_i^{(2n)} = 3\alpha^2D^4v_i^{(2n-2)} - 3\alpha^4D^2\,v_i^{(2n-4)} + \alpha^6v_i^{(2n-6)}. \tag{34d}$$

Equation (34a), which is of the form

$$y^{\text{vi}} = xy, \tag{35}$$

is the comparison equation for the present theory. This equation displays the same essential turning-point behaviour as the full equation and it thus plays the same role in the present problem as Airy's equation does in the theory of second-order equations with a simple turning point.

It is useful to note that, since an equation of the form

$$(D^6 - x)f = D^m g, \tag{36a}$$

where g satisfies the equation $(D^6 - x)\,g = 0$, has a particular solution given by

$$f = (m+1)^{-1} D^{m+1} g, \tag{36b}$$

the inhomogeneous equations (34*b*) through (34*d*) can be integrated explicitly in terms of the solutions of equation (34*a*) and their derivatives. In solving equations (34*b*) through (34*d*) we will identify only the particular solutions of the equations with the $v_i^{(2n)}(x)$, the homogeneous part of the solutions being entirely represented by the leading term $v_i^{(0)}(x)$. This procedure is necessary (see § 6) in order that the formal asymptotic solutions given by equation (33) should be asymptotic to the true solutions of equation (29).

The comparison equation (35) fortunately contains no parameters of the problem. Once a suitable set of solutions have been defined and tabulated, therefore, the solution of this problem (and of related problems having the same comparison equation) can then be carried out explicitly. In attempting to treat the present problem for large negative values of μ, it is clearly desirable, in view of the boundary conditions (30), to define the solutions of equation (35) in such a manner that three of them tend to zero for large positive values of x. A set of solutions having this property have been defined recently (Duty & Reid 1963). The required set of solutions, which we will denote by $\mathrm{A}_i(x)$ and $\mathrm{B}_i(x)$ $(i = 1, 2, 3)$, can be obtained by the method of Laplace contour integrals, in the manner described by Miller (1946) for the second-order Airy equation. For later reference, we may note that for large positive values of x, these solutions have the asymptotic behaviours

$$\begin{aligned}
\mathrm{A}_1 &\sim \tfrac{1}{2}(3\pi)^{-\frac{1}{2}} x^{-\frac{5}{12}} \exp\left(-\tfrac{6}{7}x^{\frac{7}{6}}\right),\\
\mathrm{A}_2 &\sim (3\pi)^{-\frac{1}{2}} x^{-\frac{5}{12}} \exp\left(-\tfrac{3}{7}x^{\frac{7}{6}}\right) \sin\left(\tfrac{3}{7}\sqrt{3}\, x^{\frac{7}{6}} + \tfrac{1}{3}\pi\right),\\
\mathrm{A}_3 &\sim (3\pi)^{-\frac{1}{2}} x^{-\frac{5}{12}} \exp\left(-\tfrac{3}{7}x^{\frac{7}{6}}\right) \cos\left(\tfrac{3}{7}\sqrt{3}\, x^{\frac{7}{6}} + \tfrac{1}{3}\pi\right),\\
\mathrm{B}_1 &\sim (3\pi)^{-\frac{1}{2}} x^{-\frac{5}{12}} \exp\left(+\tfrac{6}{7}x^{\frac{7}{6}}\right),\\
\mathrm{B}_2 &\sim (3\pi)^{-\frac{1}{2}} x^{-\frac{5}{12}} \exp\left(+\tfrac{3}{7}x^{\frac{7}{6}}\right) \sin\left(\tfrac{3}{7}\sqrt{3}\, x^{\frac{7}{6}} + \tfrac{1}{6}\pi\right),\\
\mathrm{B}_3 &\sim (3\pi)^{-\frac{1}{2}} x^{-\frac{5}{12}} \exp\left(+\tfrac{3}{7}x^{\frac{7}{6}}\right) \cos\left(\tfrac{3}{7}\sqrt{3}\, x^{\frac{7}{6}} + \tfrac{1}{6}\pi\right).
\end{aligned}$$

A preliminary tabulation of these solutions (together with their first five derivatives), adequate for the present purposes, has been given by Hughes & Reid (1961) for $x = -8(0{\cdot}1)+6$.

We have thus obtained an explicit asymptotic representation of the solutions appearing in the characteristic equation in terms of tabulated functions. If we now truncate the expansions (33) at $n = N$, then the resulting solution of the characteristic equation yields an Nth approximation to the curve of neutral stability. For finite negative values of μ, the required calculations are clearly somewhat lengthy; for $\mu \to -\infty$, however, the characteristic determinant is of only the third-order and detailed results can be obtained more easily. Some results for this limiting case will be given in the following section.

5. The solution for the case $\mu = -\infty$

For the case $\mu = -\infty$, the three solutions denoted by $\mathrm{B}_i(x)$ must be discarded in order to satisfy the boundary conditions at $x = +\infty$. If we then choose to identify the solutions of (34a) with the tabulated solutions $\mathrm{A}_i(x)$, then the first few coefficients of the series (33) are given by

$$\left.\begin{aligned} v_i^{(0)} &= \mathrm{A}_i, \\ v_i^{(2)} &= \alpha^2(\tfrac{3}{5}\mathrm{A}_i^{\mathrm{v}}), \\ v_i^{(4)} &= \alpha^4(\tfrac{9}{50}x\mathrm{A}_i^{\mathrm{iv}} - \tfrac{7}{25}\mathrm{A}_i'''), \\ v_i^{(6)} &= \alpha^6(\tfrac{9}{250}x^2\mathrm{A}_i''' - \tfrac{21}{125}x\mathrm{A}_i'' + \tfrac{56}{125}\mathrm{A}_i'). \end{aligned}\right\} \tag{37}$$

The characteristic determinant (32), which now reduces from the sixth- to the third-order, can be written in the form

$$\begin{vmatrix} v_i(-\lambda) \\ (D^2-\alpha^2/\lambda^2)\, v_i(-\lambda) \\ D(D^2-\alpha^2/\lambda^2)\, v_i(-\lambda) \end{vmatrix} = 0, \tag{38}$$

where $i = 1, 2, 3$.

The zero-order approximation, $v_i^{(0)}(x)$, gives the exact solution of equation (29) for the limiting case $\alpha \to 0$. The characteristic determinant (38) then takes the simple form

$$\begin{vmatrix} \mathrm{A}_i(-\lambda) \\ \mathrm{A}_i''(-\lambda) \\ \mathrm{A}_i'''(-\lambda) \end{vmatrix} = 0, \tag{39}$$

where $i = 1, 2, 3$. The least positive root of this equation is $\lambda = 2{\cdot}8378$, which implies that for the lowest mode

$$\tau \to 1482{\cdot}1/\alpha^2 \quad \text{as} \quad \alpha \to 0 \quad (\mu = -\infty). \tag{40}$$

This result is, of course, simply the limiting behaviour of τ as $\alpha \to 0$ for the first mode. By considering additional terms in the expansions (33) we obtain a sequence of approximations to the neutral curve. These results are shown in figure 1 for the first mode for various values of N. It will be seen that it is necessary to include at least three terms ($N = 2$) in the asymptotic expansion to obtain an adequate approximation to τ_c and α_c, and the inclusion of further terms causes only a slight modification of the neutral curve in the neighbourhood of the minimum point. For $N = 4$ we obtain

$$\tau_c = 1186{\cdot}4, \quad \alpha_c = 2{\cdot}0018 \quad \text{and} \quad \lambda_c = 3{\cdot}3519 \tag{41}$$

for the critical values at the onset of instability. These results compare favourably with the 'exact' values

$$\tau_c = 1178{\cdot}6, \quad \alpha_c = 2{\cdot}034 \quad \text{and} \quad \lambda_c = 3{\cdot}3622 \tag{42}$$

obtained by Harris & Reid (1964) by direct numerical integration of equation (29).

In order to determine the corresponding eigenfunction, $v(x)$, we can use either the approximate values given by (41) or the exact values given by (42). The difference between the eigenfunctions in the two cases is only detectable at large values of x. Since the exact values are available, it would seem preferable to use them. The expansion coefficients are then found to be in the ratio

$$C_1:C_2:C_3 = 1:0{\cdot}986487:0{\cdot}0202023. \tag{43}$$

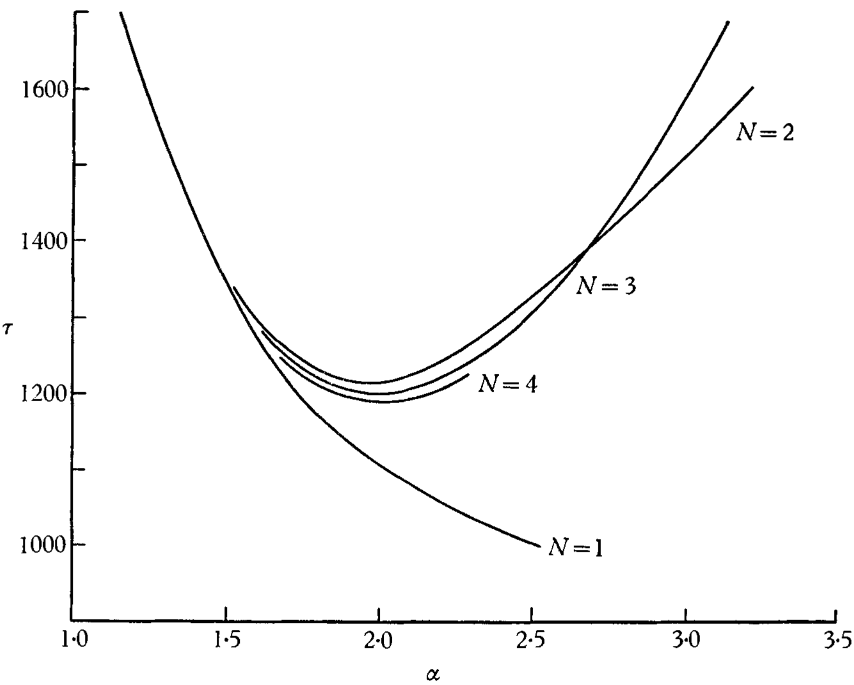

FIGURE 1. The various approximations to the curve of neutral stability for the first mode.

It is convenient to normalize the amplitude of the radial component of the perturbation velocity† to unity in order that the maximum value of the stream function occurring in the cell pattern will also be normalized to unity.

From figures 2 and 3 it is seen that, as might be expected from Rayleigh's criterion, the amplitude of the secondary motion is largest in the cell which borders the inner cylinder; this effect is also seen in the cell pattern shown in figure 4. It is also apparent that the amplitude of the secondary motion is strongly damped as one proceeds from the inner towards the outer cylinder. Since the representation (33) for $v_i(x)$ is derived for $\lambda \to \infty$ for a fixed value of x, it does not provide a satisfactory representation of the eigenfunction for a fixed value of λ as $x \to +\infty$. This difficulty can be overcome by employing the contour integral representation of the solutions of equation (29), and this alternative representation is derived in the following section. From this representation for $v(x)$ it is found that as $\mu \to -\infty$, the number of cells occurring between the cylinders becomes infinite, while the distance between the cell boundaries decreases like $x^{-\frac{1}{6}}$ as $x \to +\infty$.

† This is related to the amplitude of the azimuthal component of the perturbation velocity by the equation

$$u(x) = (\lambda^2 D^2 - \alpha^2)\, v(x).$$

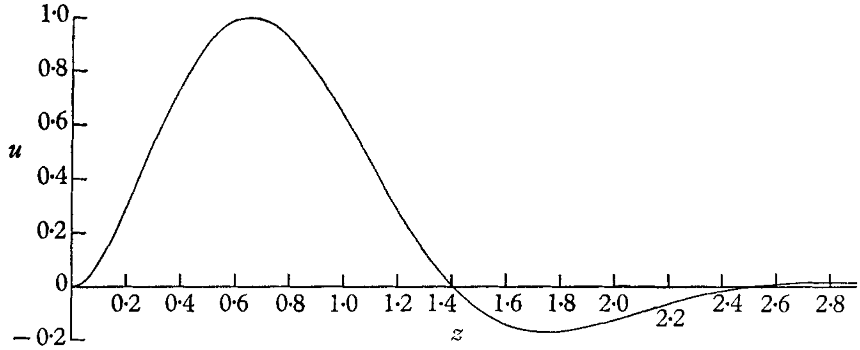

FIGURE 2. The amplitude of the radial eigenfunction u at the onset of instability for $\mu = -\infty$. The nodal surface is located at $z = 1$; $\tau_c = 1178{\cdot}6$, $\alpha_c = 2{\cdot}0337$.

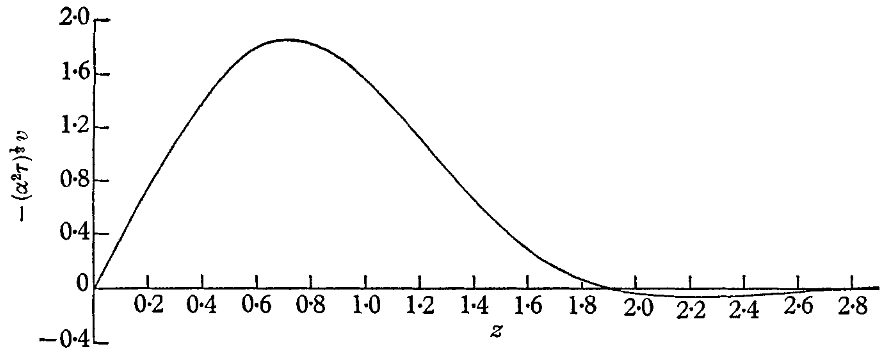

FIGURE 3. The amplitude of the azimuthal eigenfunction v at the onset of instability for $\mu = -\infty$.

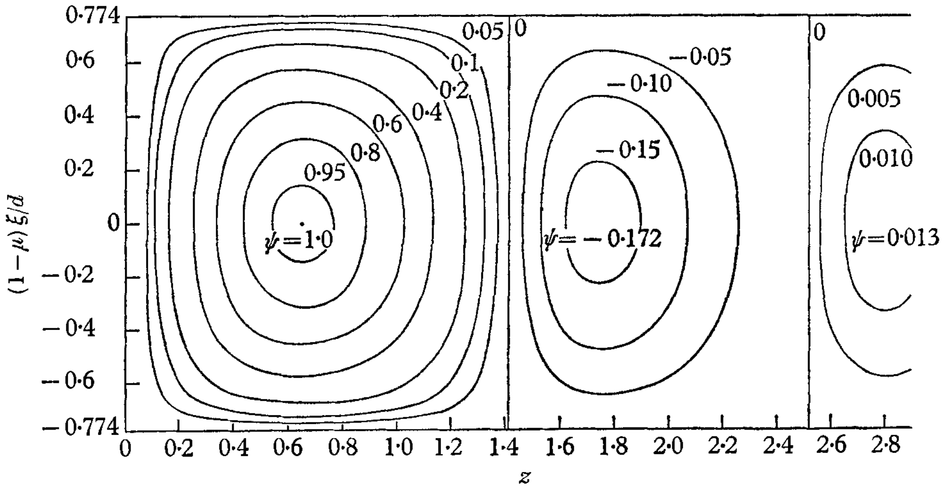

FIGURE 4. The cell pattern at the onset of instability for $\mu = -\infty$. $\psi = u(z)\cos(a\xi/d)$.

An attempt was also made to obtain approximations to the neutral curve for the second mode of instability, and the results for this case are shown in figure 5. It is seen that the representation (33), for a given order in $1/\lambda^2$, does not allow as good an approximation to the neutral curve for the second mode as it does for for the first mode. This behaviour may be traced to the fact that for the second

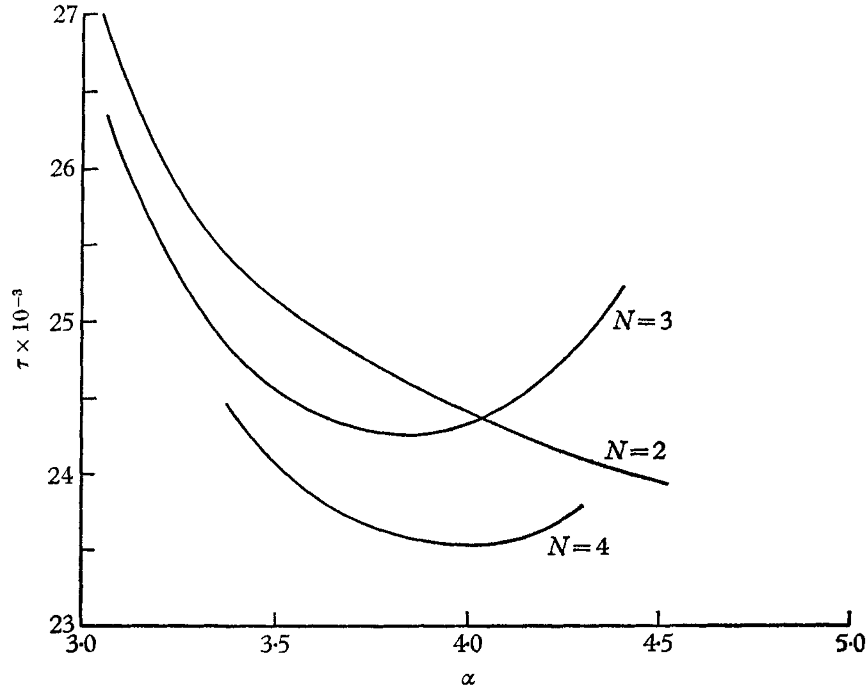

FIGURE 5. The various approximations to the curve of neutral stability for the second mode.

mode the ratio of α/λ is slightly larger than it is for the first mode. From the curve for $N = 4$, we obtain the critical values

$$\tau_c = 2{\cdot}35 \times 10^4, \quad \alpha_c = 4{\cdot}01 \quad \text{and} \quad \lambda_c = 6{\cdot}26. \tag{44}$$

These values are, of course, somewhat crude, but they do agree surprisingly well with the 'exact' values

$$\tau_c = 2{\cdot}29 \times 10^4, \quad \alpha_c = 4{\cdot}18 \quad \text{and} \quad \lambda_c = 6{\cdot}31 \tag{45}$$

obtained by Harris & Reid (1964).

6. The Laplace-integral representation

In order to obtain an adequate representation for $v(x)$ for large values of x, it is necessary to consider the Laplace contour integral solutions of equation (29). These solutions can also be used to show that the formal expansions (33) are in fact asymptotic to the true solutions of equation (29).

By the method of Laplace integrals (cf. Coddington & Levinson 1955) we find the contour integral solutions of equation (29) are of the form

$$v_i(x) = \int_{P_i} \exp[f(t)]\, G(x,t)\, dt, \tag{46}$$

where
$$f(t) = \frac{\alpha^2}{\lambda^2}\left(\tfrac{3}{5}t^5 - \frac{\alpha^2}{\lambda^2}t^3 + \frac{\alpha^4}{\lambda^4}t\right), \tag{47}$$

and
$$G(x,t) = \exp(xt - t^7/7). \tag{48}$$

The paths P_i in the complex t-plane are to be chosen so that $t^7/7$ has an infinitely positive real part at their end-points. It is possible, after investigating various combinations of paths P_i, to specify those combinations which allow the boundary conditions (30) at $x = +\infty$ to be satisfied, and it can then be shown by the method of steepest descents (cf. Jeffreys & Jeffreys 1956) that the solutions $v_i(x)$ for large positive values of x have the same ultimate behaviour as the functions denoted by $\mathrm{A}_i(x)$. The asymptotic behaviour of $v(x)$ as $x \to +\infty$ is therefore given by the expression†

$$-(\alpha_c^2 \tau_c)^{\frac{1}{3}}\, v(x) \sim 4{\cdot}7961[\mathrm{A}_1(x) + 0{\cdot}98649\mathrm{A}_2(x) + 0{\cdot}020202\mathrm{A}_3(x)]. \tag{49}$$

It is seen from equation (49) that, since $\mathrm{A}_1(x)$ is more strongly damped than either $\mathrm{A}_2(x)$ or $\mathrm{A}_3(x)$, the eigenfunction $v(x)$ closely resembles $\mathrm{A}_2(x)$ for large positive values of x, with modifications only near the zeros of $\mathrm{A}_2(x)$.

Consider now the contour integral solutions of equation (34a) which are of the form

$$v_i^{(0)}(x) = \int_{P_i} G(x,t)\, dt, \tag{50}$$

where $G(x,t)$ is defined by equation (48), and the paths P_i can be chosen to be identical with those specified for the solutions of equation (46). If we now expand the quantity $\exp[f(t)]$ which occurs in equation (46) in a Maclaurin series, and then integrate the resulting expression term by term, we obtain

$$v_i(x) = \int_{P_i} G(x,t)\, dt + \int_{P_i} f(t)\, G(x,t)\, dt + \frac{1}{2!}\int_{P_i} [f(t)]^2 G(x,t)\, dt + \ldots. \tag{51}$$

In order to compare the representation (51) for $v_i(x)$ with the one given in § 4, it is only necessary to substitute the explicit form for $f(t)$ into equation (51), and then to order the terms in ascending powers of $1/\lambda^2$. For example, the first term of equation (51) is simply $v_i^{(0)}(x)$, while the coefficient of $1/\lambda^2$, which arises from the second term of equation (51), leads to an expression of the form

$$\frac{3\alpha^2}{5\lambda^2}\int_{P_i} t^5 G(x,t)\, dt. \tag{52}$$

But this expression may be interpreted, in view of the relation (50), as

$$\frac{3\alpha^2}{5\lambda^2}\frac{d^5 v_i^{(0)}}{dx^5}, \tag{53}$$

which is, of course, identical with the term $v_i^{(2)}/\lambda^2$ given by equation (34b).

† The factor $(\alpha_c^2\tau_c)^{\frac{1}{3}}$ has been introduced simply to obtain a convenient scaling for $v(x)$.

7. Concluding remarks

The asymptotic method of solution described in this paper depends, for its simplicity, largely on the fact that we have made the small-gap approximation and have assumed the principle of exchange of stabilities to be valid. More generally, if $\eta < 1$ but μ is large and negative, the small-gap approximation can still be applied to the unstable layer of fluid adjacent to the inner cylinder but not to the outer, stable layer of fluid in which the effects of curvature remain important. If the principle of exchange of stabilities is not assumed to hold, then one obtains a 'reduced equation' of the second-order that can be solved explicitly in terms of Airy functions (Reid 1960). These inviscid solutions must differ appreciably from the solutions of the full equation not only near the turning point but also in the outer, stable portion of the flow where they in fact decay more rapidly than the viscous solutions. The analysis of equation (7) for possible overstable modes of instability is likely, therefore, to be much more intricate, and to require a knowledge of the solutions of the comparison equation $y^{\mathrm{vi}} = xy$ for complex values of x.

There would also appear to be a number of aspects of the problem which deserve further attention. For example, when the cylinders rotate in the same direction, it would appear that the relevant comparison equation must be $y^{\mathrm{vi}} = -y$ and that an analysis of the problem for $0 < \mu \leqslant 1$ along the present lines would provide a deeper understanding of the results quoted in section 1.

Another question that emerges directly from the present work concerns the behaviour of T as $a \to \infty$. For $\mu = +1$, it is known (from the exact solution of the Bénard problem) that $T \sim a^4$ as $a \to \infty$, and it would be interesting to know if a relation of the form $T \sim C(\mu)\, a^4$ continues to hold for $\mu < +1$. Neither Meksyn's work nor the present work can adequately answer this question.

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